

# The Coupling-Deformed Pointer Observables and Weak Values

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While the novel applications of weak values have recently attracted wide attention, weak measurement, the usual way to extract weak values, suffers from risky approximations and severe quantum noises. In this paper, we show the weak-value information can be obtained exactly in strong measurement with post-selections, via measuring the coupling-deformed pointer observables, i.e., the observables selected according to the coupling strength. With this approach, we keep all the advantages claimed by weak-measurement schemes and at the same time solve some widely criticized problems thereof, such as the questionable universality, systematical bias, and drastic inefficiency.

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## I. INTRODUCTION

Introduced by Aharonov, Albert and Vaidman (AAV) [1–3] about thirty years ago, the *weak value* arises in the outcome of weak measurement with postselection, which is conventionally abbreviated as *weak measurement*. Weak values play the same role in weak measurements as expectation values play in von Neumann measurements. The central task of many impressive applications of weak measurements is then to determine the weak values which could incorporate the desired information [4]. For example, weak-value tomography, to which we shall pay more attention below, has realized the direct measurements of both quantum states [5–13] and quantum dynamical processes [14]. The directness and simplicity of weak-value tomography in implementation make it perhaps the unique option for the reconstructions of high-dimensional states, and the current record is 19200-dimensional states [8]. Such kind of achievement is practically impossible for standard tomography techniques [15].

We shall briefly comment on how weak measurements perform better for many cases in the next section. Nevertheless, some severe disadvantages of weak-measurement applications have been exposed. A common trouble arises from the possible failure of weak value as the real pointer reading, especially when the weak value approaches infinity [16–19]. It makes weak-measurement tomography not universal [4, 20, 21]. This shortcoming also limits the performance of weak-value amplification [4, 22]. Moreover, weak measurements suffer from serious quantum noise, which is usually argued as the inevitable price. For example, to suppress statistical error down to the same level, weak measurements require several orders

of magnitude more samples than the standard tomography scheme [21]. Besides that, the weak limit is inaccessible since only finite (even though tiny) coupling strength can be used in experiments. Systematical errors are then introduced unavoidably and behave as inevitable bias in weak-value tomography [21].

Instead of obtaining the weak-value information approximately as in the originally proposed measurement at the weak limit, in this article, we show that one can get it exactly in the original setup but with interaction of arbitrary strength, provided that the observable read on the pointer is properly chosen as a coupling-deformed (CD) pointer observable. This is our main idea. Our method works for the whole regime of measurement strength. The exactness and the utilization of stronger measurements remove the problems of risky approximations and quantum noise in the determinations of weak values. Particularly, in the weak-value tomography, our method requires only a slight alteration to the current experimental setting and thus can be realized in experiments straightforwardly.

We arrange this paper as followed. In Sec. II, we shall introduce the standard formulas of weak values and weak measurements, and the application in quantum state tomography. In Sec. III we give our main results on coupling-deformed pointer observables and demonstrate how our method solves the mentioned problems. In Sec. IV we propose methods against possible complexities arising in the method of CD observables. Finally we discuss further implications and conclude this article in Sec. V.

## II. AHARONOV, ALBERT, AND VAIDMAN'S FORMALISM OF WEAK VALUE

In this section, we shall briefly introduce AAV's perturbation formalism of weak measurements, and the applications of weak value in quantum state tomography.

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### A. Weak measurement and weak value

The model in consideration consists of a system to be measured and a pointer. Suppose the system is initialized in the state  $\rho_{in} = |\psi_{in}\rangle\langle\psi_{in}|$ . To measure observable  $\hat{A}$ , one couples the system to the pointer initialized in the state  $|\phi_0\rangle$ , via the typical unitary time evolution operator  $U = \exp(-ig\hat{A}\otimes\hat{p})$ , where  $\hat{p}$  is defined on the pointer and  $g$  is the coupling strength (assumed to be dimensionless). After that, the system is postselected into  $\Pi_f = |\psi_f\rangle\langle\psi_f|$ . If the coupling is weak enough, the unnormalized pointer state  $\langle\psi_f|U|\psi_{in}\rangle|\phi_0\rangle$  will be [1, 2]

$$\begin{aligned} \langle\psi_f|(I - ig\hat{A}\otimes\hat{p})|\psi_{in}\rangle|\phi_0\rangle + O(g^2) \\ \approx \langle\psi_f|\psi_{in}\rangle \exp(-igA_w\hat{p})|\phi_0\rangle, \quad g \rightarrow 0. \end{aligned} \quad (1)$$

This rough derivation suggests that if  $\hat{p}$  is the momentum operator and  $\hat{q}$  is the position operator ( $[\hat{q}, \hat{p}] = i\hbar$ ), then pointer's  $\hat{q}$ -reading will be shifted by  $gA_w$ . Here

$$A_w = \frac{\langle\psi_f|\hat{A}|\psi_{in}\rangle}{\langle\psi_f|\psi_{in}\rangle} = \frac{\text{tr}(\Pi_f\hat{A}\rho_{in})}{\text{tr}(\Pi_f\rho_{in})} \quad (2)$$

is known as the *weak value*. One can determine the real and imaginary parts of weak value, respectively, by measuring two pointer observables, which are usually denoted by  $\hat{q}$  and  $\hat{p}$  [1, 2, 23]. The outcomes of such measurements, or the readings of the pointer, are denoted by  $\langle\hat{q}\rangle_f$  and  $\langle\hat{p}\rangle_f$ , respectively. The subscript “f” denotes that the expectation values are conditioned on successful postselection. The postselection is equivalent to a projective measurement. Thus the action of postselection and reading the pointer is equivalent to a measurement of  $\Pi_f \otimes \hat{q}(\hat{p})$  on the entire system. It implies that

$$\langle\hat{q}(\hat{p})\rangle_f = \frac{\langle\Pi_f \otimes \hat{q}(\hat{p})\rangle}{P_f}$$

where on the right hand side  $\langle\cdot\rangle$  stands for the average value in the coupled joint state  $U|\psi_{in}\rangle|\phi_0\rangle$ ;  $P_f = \langle\Pi_f \otimes I\rangle$  ( $I$  is the identity operator) is the probability of successful postselection. In the weak limit  $g \rightarrow 0$ ,  $P_f$  will equal the denominator of  $A_w$ ,  $\text{tr}(\Pi_f\rho_{in})$ .

### B. Applications

Here we investigate applications to determine the unknown information incorporated in the weak value. These applications are classified into two categories, “measurable complex values” and “conditioned average”, in Ref. [4]. From the definition of weak value (2), it is obvious that the numerator  $\text{tr}(\Pi_f\hat{A}\rho_{in})$  is what we are really interested in, since the denominator is directly accessible in the projective measurements.

As the natural approach to acquire weak values, weak measurements with postselections are employed in these applications. And in fact, these applications are different

in the specific selections of  $|\psi_{in}\rangle$ ,  $|\psi_f\rangle$  and  $\hat{A}$ . Thus we shall not elaborate on them but merely focus on weak-value tomography, or direct measurement of quantum states.

#### 1. Weak-value tomography

The numerator  $\text{tr}(\Pi_f\hat{A}\rho_{in})$  in the definition of  $A_w$  is actually what one wants in weak-value tomography, since it gives the wave functions [5–8] or the Kirkwood-Dirac distribution of general states [9–13]. To see it, if  $\hat{A} = |x\rangle\langle x|$  (the projector at position  $x$ ) and  $|\psi_f\rangle$  is the zero-momentum eigenstate  $|\vec{0}_p\rangle$ , then

$$(|x\rangle\langle x|)_w = \frac{\psi_{in}(x)\langle\psi_{in}|\vec{0}_p\rangle}{|\langle\vec{0}_p|\psi_{in}\rangle|^2} \quad (3)$$

where we have assumed the normalization  $\langle x|\vec{0}_p\rangle = 1$ . Thus, it is the numerator that gives the wave function  $\psi_{in}(x)$  [5].

In the experiment by Lundeen *et al.* [5], the pointer is played by photon's polarization initialized in  $|0\rangle$  ( $\sigma_z|0\rangle = |0\rangle$ ). To get the weak-value information from measurements, the pointer observables,  $\hat{p}$  and  $\hat{q}$ , are substituted with two Pauli operators,  $\hat{p} \rightarrow \sigma_x$  and  $\hat{q} \rightarrow \sigma_y$ . Explicitly, the formula for the numerator of  $(|x\rangle\langle x|)_w$  is

$$\psi_{in}(x)\langle\psi_{in}|\vec{0}_p\rangle \approx \frac{-1}{2g}\langle\Pi_f \otimes \hat{q}\rangle + \frac{i}{2g}\langle\Pi_f \otimes \hat{p}\rangle. \quad (4)$$

This equation also shows how the unknown wave function (left-hand side) is determined from the outcome of measurements (right-hand side). The formula for mixed states goes in similar ways [11, 12].

#### 2. Merits and drawbacks

The merits of weak-value tomography were highlighted in all the relevant references [5–13]. In sharply contrast to the global inversion required in the standard tomography, the most striking feature of weak-value tomography is exhibited in the direct extraction of the wave function at each spatial point. The entire wave function can be generated in real time. The second merit is the simplicity of manipulation. The observable  $\hat{A}$  is chosen from the set  $\{|x\rangle\langle x|\}_x$ . These projectors commute with each other, and correspond to one orthogonal basis of the Hilbert space. To compare, the observables required in the standard tomography do not mutually commute. In the cases in which  $\rho_{in}$  is a mixed state, the postselection should be extended to  $\{\Pi_f\}_f$  ( $\Pi_f\Pi_{f'} = \Pi_f\delta_{f,f'}$ ,  $\sum_f \Pi_f = I$ ,  $I$  is the identity operator). These postselections in weak-value tomography are compatible so that can be realized a single experimental setup. Because of these factors, weak-value tomography was successfully applied in the

estimation of high-dimension states, like 27 dimensional states in [6] and 192 dimensional states in [8].

One may wonder that why we do not directly measure the observable  $\Re(\Pi_f \hat{A})$  and  $\Im(\Pi_f \hat{A})$ . If the projective measurements of the two observables are available in the laboratory, perhaps that would be better. However, here they are written in the form of  $|x\rangle\langle\vec{0}_p| + |\vec{0}_p\rangle\langle x|$  and  $i|x\rangle\langle\vec{0}_p| - i|\vec{0}_p\rangle\langle x|$ . It seems impossible to directly implement the measurements of them with the current capability. Moreover, in the context of tomography, measuring the complete set of these observables is just another version of the standard tomography.

The drawbacks of weak-measurement schemes come from the approximations and the weakness. The error of the approximations used in Eq. (1) is drastic when  $\text{tr}(\Pi_f \rho_{in})$  is close to zero [16–18]. To avoid this trouble one should know enough information about  $\rho_{in}$ , which seems impossible since  $\rho_{in}$  is preassumed as unknown. It makes the method not universal [4, 20, 21]. Other problems, such as bias and inefficiency, have been mentioned above. These drawbacks are rooted in the original formalism of weak measurements and weak value and thus cannot be solved trivially.

### III. COUPLING-DEFORMED POINTER OBSERVABLE

In this section, we will introduce the concept of the coupling-deformed pointer observable, which permits us to extract the weak-value information exactly with measurements of any strength. See Fig. 1 for an overview.

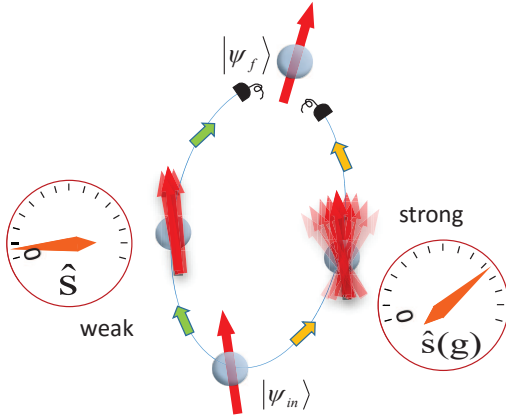


Figure 1: The extraction of weak-value information starts from the initial state preparation of  $|\psi_{in}\rangle$ , and ends at the postselection of the system onto  $|\psi_f\rangle$ . AAV's formalism requires the measurement to be as weak as possible (left path in the figure). While we show that we can get the weak-value information using measurements at an arbitrary strength, say  $g$ , if we replace the pointer observable with the coupling-deformed pointer observable  $\hat{s}(g)$  (right path).

### A. A nonperturbative framework

To go beyond the first-order perturbation [16, 17, 19, 24–27], the popular method is to keep more terms of the series expansion

$$U = \sum_{n=0}^{\infty} \frac{(-ig)^n}{n!} \hat{A}^n \otimes \hat{p}^n. \quad (5)$$

However, expressions written with a long summation are too complicate to be used for further analysis and applications. For our purpose, a simple and fast nonperturbative approach is posteriorly proved to be much better.

Suppose  $\hat{A} = \sum_{m=1} a_m |a_m\rangle\langle a_m|$  with  $\langle a_m | a_n \rangle = \delta_{mn}$  and  $a_m \neq 0$ . Then one has  $|\psi_{in}\rangle = \sum_{m=1} c_m |a_m\rangle + c_0 |\psi^\perp\rangle$  where  $|\psi^\perp\rangle$  is the component of  $|\psi_{in}\rangle$  in the null space of  $\hat{A}$ . After the coupling  $U = \exp(-ig\hat{A} \otimes \hat{p})$ , the overall state becomes

$$c_0 |\psi^\perp\rangle |\phi_0\rangle + \sum_{m=1} c_m |a_m\rangle |\phi_m(g)\rangle, \quad (6)$$

where  $|\phi_m(g)\rangle = \exp(-iga_m \hat{p}) |\phi_0\rangle$ . We denote  $|\psi^\perp\rangle$  as  $|a_0\rangle$  for convenience. Suppose  $\hat{s} \in \{\hat{p}, \hat{q}\}$ , momentum or position when the pointer is a continuous-variable system, or Pauli operators when the pointer is a qubit system, or other observables of the pointer accordingly. Then the expectation value of  $\Pi_f \otimes \hat{s}$  for the given coupling strength  $g$  can be written in a compact form as

$$\langle \Pi_f \otimes \hat{s} \rangle_g = g \sum_{m,n=0} (\rho_{in})_{mn} (\Pi_f)_{nm} Q_{nm}(g, \hat{s}), \quad (7a)$$

$$Q_{nm}(g, \hat{s}) = \frac{1}{g} \langle \phi_n(g) | \hat{s} | \phi_m(g) \rangle. \quad (7b)$$

where  $(\rho_{in})_{mn} = \langle a_m | \rho_{in} | a_n \rangle$ ,  $(\Pi_f)_{nm} = \langle a_n | \Pi_f | a_m \rangle$ . Additionally, when the input state  $\rho_{in}$  is not pure, the definition of  $|a_0\rangle$  becomes ambiguous. In this case, suppose  $|\psi_f\rangle = \sum_{m=1} \beta_m |a_m\rangle + \beta_0 |\psi_f^\perp\rangle$ . Then  $|a_0\rangle$  could be redefined as  $|\psi_f^\perp\rangle$ . It is not complicated to check that this redefinition will not change the value of the right-hand side of Eq. (7a), which then turns out to be applicable to situations with  $\rho_{in}$  being a mixed state.

### B. A stronger sufficient condition for the extraction of weak-value information

Within the above nonperturbation framework, let us investigate how the numerator of the weak value,  $\text{tr}(\Pi_f \hat{A} \rho_{in})$ , arises at the weak limit. The result will hint at a sufficient condition for the extraction of weak-value information that is stronger than the weak limit.

Under the condition that  $\langle \phi_0 | \hat{s} | \phi_0 \rangle = 0$ , the  $Q$ -matrix elements read in the weak limit as

$$\begin{aligned} \lim_{g \rightarrow 0} Q_{nm}(g, \hat{s}) &= ia_n \langle \phi_0 | \hat{p} \hat{s} | \phi_0 \rangle - ia_m \langle \phi_0 | \hat{s} \hat{p} | \phi_0 \rangle \\ &\equiv Q_{nm}(0, \hat{s}) \end{aligned} \quad (8)$$

Substituting Eq. (8) into Eq. (7a), one gets

$$\begin{aligned} \langle \Pi_f \otimes \hat{s} \rangle_{g \rightarrow 0} &= -ig \langle \phi_0 | [\hat{s}, \hat{p}]_- | \phi_0 \rangle \Re [\text{tr}(\Pi_f \hat{A} \rho_{in})] \\ &+ g \langle \phi_0 | [\hat{s}, \hat{p}]_+ | \phi_0 \rangle \Im [\text{tr}(\Pi_f \hat{A} \rho_{in})] \end{aligned} \quad (9)$$

Here  $\Re$  and  $\Im$  mean the real and imaginary parts, respectively, and  $[\hat{s}, \hat{p}]_{\pm} = \hat{s}\hat{p} \pm \hat{p}\hat{s}$ . That means, for the real part of weak-value numerator, one should read a point observable  $\hat{q}$ , i.e.,  $\hat{s} \rightarrow \hat{q}$ , which satisfies  $\langle \phi_0 | [\hat{q}, \hat{p}]_+ | \phi_0 \rangle = 0$ , and for the imaginary part, one can read  $\hat{s} \rightarrow \hat{p}$ , which obviously satisfies  $\langle \phi_0 | [\hat{p}, \hat{p}]_- | \phi_0 \rangle = 0$ . Explicitly,

$$\begin{aligned} \Re [\text{tr}(\Pi_f \hat{A} \rho_{in})] &= \frac{i}{g \langle \phi_0 | [\hat{q}, \hat{p}]_- | \phi_0 \rangle} \langle \Pi_f \otimes \hat{q} \rangle_{g \rightarrow 0}, \\ \Im [\text{tr}(\Pi_f \hat{A} \rho_{in})] &= \frac{1}{g \langle \phi_0 | [\hat{p}, \hat{p}]_+ | \phi_0 \rangle} \langle \Pi_f \otimes \hat{p} \rangle_{g \rightarrow 0}. \end{aligned} \quad (10)$$

These two expressions verify the sufficiency of weak limit, while the derivation for Eq. (10) illuminates a wider sufficient condition: the elements of the  $Q$  matrix,  $Q(g, \hat{s})$ , have the special type that  $Q_{mn} = ca_m + c^* a_n$  with  $c$  being a constant. To reach this type of  $Q$  matrix, weak limit is sufficient, but not necessary.

### C. Coupling-deformed pointer observables

In the weak limit, Eqs. (8) and (9) suggest that the specific selection of  $\hat{s}$  is not important: we just need to make sure that the factor  $\langle \phi_0 | \hat{p} \hat{s} | \phi_0 \rangle$  is real or imaginary, while to go beyond the weak regime, the key observation is that we can exploit the untapped freedom of choosing a proper  $\hat{s}$ .

Let us denote the modified pointer observable as  $\hat{s}(g)$ , which we call the *coupling-deformed* (CD) observable hereafter. We can get the weak-value numerator if the selection of  $\hat{s}(g)$  makes the corresponding  $Q$  matrix,  $Q[g, \hat{s}(g)]$ , satisfy the relation

$$Q[g, \hat{s}(g)] = \eta Q(0, \hat{s}) \quad (11)$$

where  $\eta$  is a proportionality constant, which may or may not depend on  $g$ . To see it, from Eqs. (7a) and (11) we have

$$\langle \Pi_f \otimes \hat{s}(g) \rangle_g = \eta \langle \Pi_f \otimes \hat{s} \rangle_{g \rightarrow 0}. \quad (12)$$

Then, as an alternative to Eq. (10), if the corresponding  $\hat{q}(g)$  and  $\hat{p}(g)$  exist we have

$$\begin{aligned} \Re [\text{tr}(\Pi_f \hat{A} \rho_{in})] &= \frac{i}{\eta g \langle \phi_0 | [\hat{q}, \hat{p}]_- | \phi_0 \rangle} \langle \Pi_f \otimes \hat{q}(g) \rangle_g, \\ \Im [\text{tr}(\Pi_f \hat{A} \rho_{in})] &= \frac{1}{\eta g \langle \phi_0 | [\hat{p}, \hat{p}]_+ | \phi_0 \rangle} \langle \Pi_f \otimes \hat{p}(g) \rangle_g. \end{aligned} \quad (13)$$

Therefore, we can obtain the weak-value information exactly in strong measurements of any strength  $g$ , via reading the CD observables  $\hat{q}(g)$  and  $\hat{p}(g)$ .

Then how can we identify the desired CD observable  $\hat{s}(g)$  ( $\hat{s} = \hat{q}$  or  $\hat{p}$ )? At first, one can choose an arbitrary orthonormal basis  $\{|\tilde{u}\rangle\}$  of the space spanned by  $\{|\phi_m(g)\rangle\}$ , and define the matrix  $S(g)$  with elements given by  $S_{um} = \langle \tilde{u} | \phi_m(g) \rangle$ . Suppose  $\{|\phi_m(g)\rangle\}$  are linearly independent (we will discuss the other case later); the matrix  $S(g)$  will have a well-defined inversion  $S(g)^{-1}$ . One can define a matrix  $\tilde{Q}(g, \hat{s})$  by

$$\tilde{Q}(g, \hat{s}) = \eta [S^\dagger(g)]^{-1} Q(0, \hat{s}) S(g)^{-1}, \quad (14)$$

where  $Q(0, \hat{s})$  is given by (8). Then, the CD observable can be chosen as

$$\hat{s}(g) = g \sum_{u,v} (\tilde{Q}(g, \hat{s}))_{uv} |\tilde{u}\rangle \langle \tilde{v}|. \quad (15)$$

It is straightforward to show that the choice of CD observable according to (15) ensures the requirement (11), and therefore also ensures (13). So one can choose  $\hat{q}(g)$  [or  $\hat{p}(g)$ ] according to (15), and then retrieve the real (or imaginary) part of the weak-value information  $\text{tr}(\Pi_f \hat{A} \rho_{in})$  from (13) by measuring the expectation value of  $\Pi_f \otimes \hat{q}(g)$ , or  $\Pi_f \otimes \hat{p}(g)$ , with any strength  $g$ .

Here are some remarks. Equation (15) fixes the effective parts of  $\hat{s}(g)$ ; one can add irrelevant terms living outside of the space spanned by  $\{|\phi_m(g)\rangle\}$ . The proportionality constant  $\eta$  in Eq. (14) is irrelevant to the statistics of the outcomes of measuring  $\hat{s}(g)$  and thus can be fixed by the convenience. The choice of  $\hat{s}(g)$  is independent of the initial system state  $\rho_{in}$  and thus can be accomplished beforehand in the step of pointer calibration.

### D. Example: Weak-value tomography

Now we show how to use our method of CD observables to extract weak-value information. Let us revisit the experiment of Lundeen *et al.* [5], where a photon's spatial wave function is coupled to polarization (pointer) via  $U = \exp(-ig|x\rangle\langle x| \otimes \sigma_x)$  (we have  $\hat{p} \rightarrow \sigma_x$ ). Since  $\hat{A} = |x\rangle\langle x|$  is a rank-1 projector with  $a_0 = 0$  and  $a_1 = 1$ , there are only two relevant pointer states,  $|\phi_0\rangle = |0\rangle$  and  $|\phi_1(g)\rangle = \exp(-ig\sigma_x)|\phi_0\rangle = \cos(g)|0\rangle - i\sin(g)|1\rangle$  ( $\sigma_z|1\rangle = -|1\rangle$ ). Thus the  $Q$  matrices will be 2 dimensional. In order to retrieve the real part of the weak value, one has  $\hat{q} \rightarrow \sigma_y$ . According to Eq. (8),

$$Q(0, \hat{q}) = - \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}. \quad (16)$$

Now a convenient choice of the orthogonal basis of the space spanned by  $\{|\phi_0\rangle, |\phi_1(g)\rangle\}$  is  $\{|0\rangle, |1\rangle\}$ , with which the matrix  $S(g)$  introduced above can be written as

$$S(g) = - \begin{pmatrix} 1 & \cos(g) \\ 0 & -i\sin(g) \end{pmatrix}. \quad (17)$$

Then the matrix  $\tilde{Q}(g, \hat{q})$  given by Eq. (14) is

$$\tilde{Q}(g, \hat{q}) = \frac{\eta}{\sin(g)} \begin{pmatrix} 0 & -i \\ i & -2 \tan(\frac{g}{2}) \end{pmatrix}. \quad (18)$$

Fixing  $\eta = \sin(g)/g$  for convenience, from Eq. (15) we obtain the CD observable

$$\hat{q}(g) = \sigma_y - \tan(\frac{g}{2})(I - \sigma_z), \quad g \in [0, \frac{\pi}{2}]. \quad (19)$$

According to (13), we have

$$\Re[\text{tr}(\Pi_f \hat{A} \rho_{in})] = \frac{-1}{2 \sin(g)} \langle \Pi_f \otimes \hat{q}(g) \rangle_g. \quad (20)$$

The case of a strong limit achieved at  $g = \frac{1}{2}\pi$  is also studied in Ref. [28], and cases of arbitrary strength in Ref. [29]. Their methods are based on Eq. (5). An analytical result is possible because Lundeen's setup involves only a simple  $\hat{A}$  (projector) and simple pointer (qubit), while our result originates from a more general approach with CD observable.

### E. Resolution of the drawbacks

Here we explain how the shortcomings of the previous weak-value tomography listed in Ref. [21] are overcome by our approach of CD observables. A more elaborate study is reported in an independent work with other colleagues [30].

When the tiny but finite coupling strength is fixed, the pointer reading will deviate from the  $gA_w$  when  $|\psi_{in}\rangle$  tends to be orthogonal to  $|\psi_f\rangle$  [16, 18]. Our method will not face such possible failure, because the formalism is exact and universally valid.

The formalism of the weak-measurement scheme is exact only in the weak limit, while the real experiments need a finite interaction strength. This discrepancy causes the bias, the deviation between the expectation of the estimator (the reconstructed states) and the real value of the estimated state. In weak-value tomography, this bias is confirmed to be very robust [21], i.e., hard to be removed, while here it automatically disappears (we presume the perfect implementation) because of the exactness of our formalism in the whole range of coupling strength.

The most important improvement is the reduction of quantum noise. With our formalism, weak-value tomography can be implemented using stronger interactions. The efficiency of information extraction will be superior to a significant extent.

To verify the argument, we performed a numerical simulation of three tomography methods, the traditional weak-measurement schemes, the standard tomography, and our method with CD pointer observables. We randomly select a qubit state, simulate  $N$  times of quantum measurements, and then calculate the trace distance between the real state and the reconstructed state.

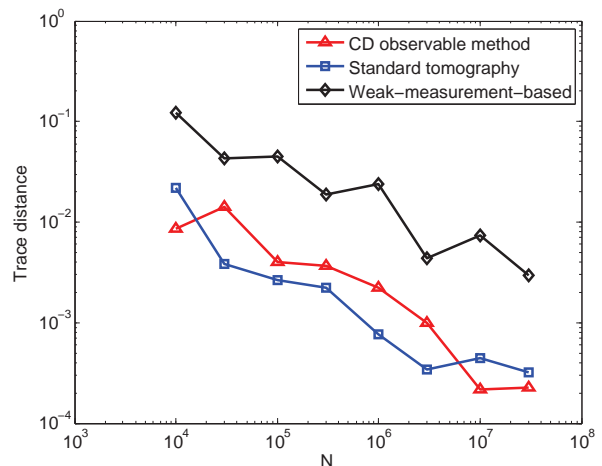


Figure 2: Trace distance (between the real and the reconstructed states) against the times of repetitions (sample size,  $N$ ). The weak-value tomography based on weak measurements is simulated with the strength  $g = 0.1$ , our method with the CD observables is simulated with the strength  $g = \pi/2$ .

The numerical result is illustrated in Fig. 2, which gives curves of reconstruction error (trace distance) against the times of repetition (sample size). It shows that, to reach the same level of precision, the method based on weak measurements needs many more samples than the other two, while the sample size used in our method with the CD observables is comparable with that of the standard tomography. Thus, the problem of resource consumption is cured. A more thorough comparison between the three is reported in Ref. [30].

### F. The $g$ -invariant CD observables

There are situations when the CD observable  $\hat{s}(g)$  is independent of  $g$ . Such  $g$ -invariant CD observables could simplify the experimental implementation. The  $g$ -invariant observables will also be necessary for applications when the coupling constant is either unknown or uncertain with inevitable errors. Below we give examples of  $g$ -invariant observables that are easy to measure.

#### 1. $\hat{A}$ is a projector

We consider again the experiment of Lundeen *et al.* [5] in which one can obtain the imaginary part of the weak value by measuring  $\hat{p} \rightarrow \sigma_x$ . We have

$$Q(g, \sigma_x) = \frac{\sin(g)}{g} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{\sin(g)}{g} Q(0, \sigma_x). \quad (21)$$

Thus Eq. (11) is fulfilled, and  $\sigma_x$  is certainly a  $g$ -invariant CD observable in this case. If the pointer is replaced by a continuous-variable system and initialized in a Gaussian

state with standard deviation  $\Delta$ ,

$$\langle q|\phi_0\rangle = \frac{1}{(2\pi\Delta)^{1/4}} \exp(-\frac{q^2}{4\Delta^2}), \quad (22)$$

which is very common in reported experiments on weak measurements [4], we will have  $(\hat{p} = -i\frac{\partial}{\partial q})$

$$Q(g, \hat{p}) = \frac{1}{4\Delta^2} e^{-\frac{g^2}{8\Delta^2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = e^{-\frac{g^2}{8\Delta^2}} Q(0, \hat{p}). \quad (23)$$

The CD observable is found to be the original observable  $\hat{p}$ , which is of course  $g$ -invariant.

## 2. $\hat{A}$ is effectively a Pauli operator

Another case of 2-dimensional  $Q$  matrix is when  $\hat{A}$  has only two nonzero eigenvalues while  $|\psi_f\rangle$  resides in the support set of  $\hat{A}$ .

Suppose that  $\hat{A} = a_1\Pi_1 + a_2\Pi_2$  where  $a_{1(2)} \neq 0$  and  $\Pi_{1(2)}$  is the projector onto the eigenspace. Then the formalism of the system with variables combination  $(g, \hat{A}, |\phi_0\rangle)$  is equivalent to that of another system where the experimental variables are  $(g_{eff}, \hat{A}_{eff}, |\phi'_0\rangle)$ :

$$g_{eff} = \frac{a_1 - a_2}{2}g; \quad \hat{A}_{eff} = \Pi_1 - \Pi_2; \quad (24)$$

$$|\phi'_0\rangle = \exp(-ig\frac{a_1 + a_2}{2}\hat{p})|\phi_0\rangle.$$

The equivalence is that, the elements of  $Q$ -matrices can be calculated with the pointer states given by

$$|\phi_{\pm}\rangle = \exp(\pm ig_{eff}\hat{p})|\phi'_0\rangle. \quad (25)$$

Thus we just need to consider operators in the form of  $\hat{A}_{eff}$ , a generalization of Pauli operator  $\sigma_z$ . But note that if  $(\Pi_1 + \Pi_2)|\psi_f\rangle \neq |\psi_f\rangle$ ,  $|\phi'_0\rangle$  should be included so that  $Q$  matrices become three dimensional.

The investigated two-dimensional  $Q$  matrices are summarized in Tab. I, where those support  $g$ -invariant pointer observables are marked. Situations when  $\hat{A}$  is the generalized Pauli matrix are discussed in Ref. [14]. In the three cases marked in Tab. I, if the pointer is played by a qubit in state  $|0\rangle$ , then the pointer observables can be chosen as Pauli operators  $\hat{p} \rightarrow \sigma_x$  and  $\hat{q} \rightarrow \sigma_y$ , while for the continuous-variable pointers and the initial state  $|\phi_0\rangle$  satisfying some conditions to be given below, the CD observables will be  $g$ -invariant.

Suppose  $\hat{q}$  is the position operator and  $\hat{p}$  is the momentum operator. Particularly, in the right column of Table I, the diagonal elements of the  $Q$  matrices (for the imaginary parts of the weak-value numerator) are derived as

$$\langle \phi_0 | e^{igam\hat{p}} \hat{p} e^{-igam\hat{p}} | \phi_0 \rangle = \langle \phi_0 | \hat{p} | \phi_0 \rangle = 0, \quad (26)$$

provided that the initial reading of the pointer is zero. The two off-diagonal terms must be complex conjugate

$\hat{A}$	$Q(0, \hat{q})$	$Q(0, \hat{p})$
$\Pi$	$-\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$	$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}_{\checkmark}$
$\Pi_1 - \Pi_2$	$2\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\checkmark}$	$2\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}_{\checkmark}$

Table I: The 2-dimensional  $Q$  matrices,  $\hat{q}$  and  $\hat{p}$  are the pointer observables for the real and imaginary parts of weak-value information, respectively. In cases marked with “ $\checkmark$ ”, the CD observables are  $g$ -invariant when the pointers are initialized in proper states.

since  $Q$  matrices are Hermitian by definition. Therefore, the desired  $Q$  matrices are ensured if

$$\langle \phi_0 | \hat{p} \exp(-ig(a_n - a_m)\hat{p}) | \phi_0 \rangle \quad (27)$$

is a pure imaginary number. It is sufficient to require the wave function of  $|\phi_0\rangle$  in momentum representation, i.e.,  $\phi_0(p)$  to have even parity.

For the  $g$ -invariant CD observable in the left column, being a continuous variable state,  $|\phi_0\rangle$  should satisfy the condition that

$$\langle \phi_0 | [\hat{q}, e^{i2g\hat{p}}]_+ | \phi_0 \rangle = 0. \quad (28)$$

It degenerates to the condition  $\langle \phi_0 | [\hat{q}, \hat{p}]_+ | \phi_0 \rangle = 0$  in the weak limit.

## IV. METHODS AGAINST POOR POINTERS

Our general method with CD observables requires the linear independence of  $\{|\phi_m(g)\rangle\}$ . But if it cannot be fulfilled due to a pointer with limited dimensions of Hilbert space, or if the CD observables are either hard to calculate or hard to measure in practice, do we have methods to circumvent these troubles?

One strategy is to measure another observable, say  $\tilde{A}$ , which supports 2-dimensional  $Q$  matrices, and of which the weak value has the same information of  $A_w$ . One construction is like this: define

$$|\psi_A\rangle = \mathcal{N} \sum_m a_m \langle a_m | \psi_f \rangle |a_m\rangle, \quad (29)$$

where  $\mathcal{N} = 1/\sqrt{\langle \psi | \hat{A}^2 | \psi_f \rangle}$  is the normalization factor. Then we have

$$\Pi_f \hat{A} \propto \Pi_f |\psi_A\rangle \langle \psi_A|. \quad (30)$$

where the proportionality factor is  $\frac{\langle \psi_f | \hat{A}^2 | \psi_f \rangle}{\langle \psi | \hat{A} | \psi_f \rangle}$ . It implies that  $\tilde{A}$  can be the rank-1 projector,  $|\psi_A\rangle \langle \psi_A|$ , provided that  $\langle \psi_f | \psi_A \rangle \neq 0$ . Otherwise  $\Pi_f |\psi_A\rangle \langle \psi_A|$  is trivially zero. In this case,  $\tilde{A}$  can be selected as the generalized Pauli matrix  $|\psi_A\rangle \langle \psi_f| + |\psi_f\rangle \langle \psi_A|$  which allows 2-dimensional  $Q$  matrices. Then even a poor pointer with

only 2-dimensional Hilbert space can be used to extract weak-value information.

Another strategy is to modify the postselection. Consider the following equality

$$\text{tr}(\Pi_f \hat{A} \rho_{in}) = [\text{tr}(\hat{A} \Pi_f \rho_{in})]^*. \quad (31)$$

That is, we can use  $\tilde{A} = \Pi_f$  and replace the original postselection with a projective measurement of  $\hat{A}$ , i.e., project the system onto  $\{|a_m\rangle\}_m$  and multiply the pointer readings by the corresponding  $a_m$  (when the system is projected onto  $|a_m\rangle$ ). Then Eq. (31) shows that the resulted value is just the complex conjugation of the original weak value. We remark here that since  $\hat{A}$  is a projector here, the  $Q$  matrices are 2 dimensional. Meanwhile, the CD observables are irrelevant to the postselected state. Therefore, only a fixed setting of pointer system is needed, although the postselection is performed onto a set of orthogonal states.

## V. DISCUSSION AND CONCLUSIONS

We would like to discuss more the experimental realizations. Given a weak-measurement scheme, our main result implies that, the measurement can be strengthened if the CD observables can be read on the pointer. That means the feasibility of our method relies on the presumption that the pointer can be operated conveniently. Otherwise, we have to measure  $\tilde{A}$  instead of the original  $\hat{A}$ , as discussed in Sec. IV, while in the important example of weak-value tomography, our method is very easy to realize within Lundeen's setup [5], where the pointer is played by the polarization freedom. Operators of any direction (in the Bloch sphere) can be simply measured. Meanwhile, the coupling between spatial wave function and polarization (the pointer) is realized with a rectangular sliver of a half-wave plate. The coupling strength is determined by the angle shift on the photons' polar-

ization. To implement stronger measurements, we just need to tune larger this angle. Therefore, everything can be done easily.

Our result also has other implications. For example, the accessibility of weak-value information is often attributed to the negligible disturbance caused by weak measurements in the literature; see, e.g., Refs. [5, 31]. However, our results show that such interpretation is unnecessary. We hope this work could stimulate more sparks on theories and applications of "weak" measurements.

To summarize, we have developed a nonperturbative approach to retrieve weak-value information in measurements with post-selections. Here the system-pointer coupling strength can be of any finite value, not necessarily small. To retain the original form of the weak-value, we can slightly modify the current weak-measurement scheme, and read the CD observables on the pointers instead. We also studied situations when such a modification is unnecessary, namely, the CD observables are  $g$ -invariant. This is meaningful for simplifying the experiments. Thus, while keeping the advantages of current weak-measurement and weak-value motivated applications, our method eliminates main problems therein, such as inefficiency, bias, and problematic universality in the current weak-measurement tomography scheme, without introducing much complexity in experimental implementations.

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- [1] Y. Aharonov, D. Z. Albert, and L. Vaidman, *How the Result of a Measurement of a Component of the Spin of a Spin-1/2 Particle Can Turn out to be 100*, Phys. Rev. Lett. **60**, 1351 (1988).
  - [2] Y. Aharonov and L. Vaidman, *Properties of a Quantum System during the Time Interval Between Two Measurements*, Phys. Rev. A **41**, 11 (1990).
  - [3] Y. Aharonov, S. Popescu, and J. Tollaksen, *A Time-Symmetric Formulation of Quantum Mechanics*, Phys. Today **63**, 27 (2010).
  - [4] J. Dressel, M. Malik, F. M. Miatto, A. N. Jordan, and R. W. Boyd, *Colloquium: Understanding Quantum Weak Values: Basics and Applications*, Rev. Mod. Phys. **86**, 307 (2014).
  - [5] J. S. Lundeen, B. Sutherland, A. Patel, C. Stewart, and C. Bamber, *Direct Measurement of the Quantum Wavefunction*, Nature **474**, 188 (2011).
  - [6] M. Malik, M. Mirhosseini, M. P. J. Lavery, J. Leach, M. J. Padgett, and R. W. Boyd, *Direct Measurement of a 27-dimensional Orbital-Angular-Moment State Vector*, Nature Commun. **5**, 3115 (2014).
  - [7] H. Kobayashi, K. Nonaka, and Y. Shikano, *Extracting Joint Weak Values from Two-dimensional Spatial Displacements*, Phys. Rev. A **89**, 053816 (2014).
  - [8] M. Mirhosseini, O. S. Magaña-Loaiza, S. M. H. Rafsanjani, and R. W. Boyd, *Compressive Direct Measurement of the Quantum Wave Function*, Phys. Rev. Lett. **113**, 090402 (2014).
  - [9] J. Z. Salvail, M. Agnew, A. S. Johnson, E. Bolduc, J. Leach, and R. W. Boyd, *Full Characterization of Polarization States of Light via Direct Measurement*, Nature Photonics **7**, 316 (2013).
  - [10] C. Bamber and J. S. Lundeen, *Observing Dirac's Classical Phase Space Analog to the Quantum State*, Phys.

- Rev. Lett. **112**, 070405 (2014).
- [11] J. S. Lundeen and C. Bamber, *Procedure for Dirac Measurement of General Quantum State Using Weak Measurement*, Phys. Rev. Lett. **108**, 070402 (2012).
  - [12] S. Wu, *State Tomography via Weak Measurement*, Sci. Rep. **3**, 1193 (2013);
  - [13] A. Di Lorenzo, *Sequential Measurement of Conjugate Variables as an Alternative Quantum State Tomography*, Phys. Rev. Lett. **110**, 010404 (2013).
  - [14] Y.-X. Zhang, S. Wu, and Z.-B. Chen, arXiv:1309.5780.
  - [15] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).
  - [16] S. Wu and Y. Li, *Weak Measurements beyond the Aharonov-Albert-Vaidman Formalism*, Phys. Rev. A **83**, 052106 (2011).
  - [17] A. Di Lorenzo, *Full Counting Statistics of Weak-Value Measurement*, Phys. Rev. A, **85**, 032106 (2012).
  - [18] S. Pang, S. Wu, and Z.-B. Chen, *Weak Measurement with Orthogonal Preselection and Postselection*, Phys. Rev. A **86**, 022112 (2012).
  - [19] X. Zhu *et al.*, *Quantum Measurements with Preselection and Postselection*, Phys. Rev. A **84**, 052111 (2011).
  - [20] E. Haapasalo, P. Lahti, and J. Schultz, *Weak Versus Approximate Values in Quantum State Determination*, Phys. Rev. A **84**, 052107 (2011).
  - [21] L. Maccone and C. C. Rusconi, *State Estimation: A Comparison between Direct State Measurement and Tomography*, Phys. Rev. A **89**, 022122 (2014).
  - [22] O. Hosten and P. Kwiat, *Observation of the Spin Hall Effect of Light via Weak Measurements*, Science **319**, 787 (2008); A. Feizpour, X. Xingxing, and A. M. Steinberg, *Amplifying Single-Photon Nonlinearity Using Weak Measurements*, Phys. Rev. Lett. **107**, 133603 (2011); N. Brunner and C. Simon, *Measuring Small Longitudinal Phase Shifts: Weak Measurements or Standard Interferometry?* Phys. Rev. Lett. **105**, 010405 (2010); see more in the references of Ref. [4].
  - [23] R. Jozsa, *Complex Weak Values in Quantum Measurements*, Phys. Rev. A **76**, 044103 (2007).
  - [24] J. Dressel, S. Agarwal, and A. N. Jordan, *Contextual Values of Observables in Quantum Measurements*, Phys. Rev. Lett. **104**, 240401 (2010).
  - [25] J. Dressel and A. N. Jordan, *Contextual Values Approach to the Generalized Measurement of Observables*, Phys. Rev. A **85**, 022123 (2012).
  - [26] Abraham G. Kofman, Sahel Ashhab, and Franco Nori, *Nonperturbative Theory of Weak Pre- and Post-Selected Measurements*, Phys. Rep. **520**, 43 (2012).
  - [27] J. Dressel and A. N. Jordan, *Weak Values are Universal in von Neumann Measurements*, Phys. Rev. Lett. **109**, 230402 (2012).
  - [28] P. Zou, Z.-M. Zhang, and W. Song, *Direct measurement of general quantum states using strong measurement*, Phys. Rev. A **91**, 052109 (2015).
  - [29] G. Vallone and D. Dequal, *Direct measurement of the quantum wavefunction by strong measurements*, Phys. Rev. Lett. **116**, 040502 (2016).
  - [30] X. Zhu, Y.-X. Zhang, S. Wu, *Unbiased state reconstruction using modified weak measurement*, arXiv: 1512.03652.
  - [31] S. Kocsis *et al.*, *Observing the Average Trajectories of Single Photons in a Two-Slit Interferometer*, Science **332**, 1170 (2011).